THE ROAD-COLOURING PROBLEM†

BY

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ABSTRACT

Let G be a finite directed graph which is irreducible and aperiodic. Assume each vertex of G leads to at least two other vertices, and assume G has a cycle of prime length which is a proper subset of G. Then there exist two functions $r: G \to G$ and $b: G \to G$ such that if $r(x) = y$ and $b(x) = z$ then $x \to y$ and $x \to z$ in G and $y \neq z$ and such that some composition of r's and b's is a constant function.

1. Introduction

Let G be a directed graph with finite vertex set S. Assume throughout that G is *irreducible*, that is, for all proper subsets U of S, there are vertices $x \in U$ and $y \in S - U$ such that the edge $x \to y$ is in the graph G. (We use the notation "$x \to y$ in G" since at times other graphs with vertex set S will be considered.) Assume also that G is *aperiodic*, that is, S cannot be partitioned into $n > 1$ subsets $S_1, S_2, \cdots, S_n = S_0$ in such a way that $x \to y$ in G and $x \in S_i$ together imply $y \in S_{i-1}$, $i = 1, 2, \cdots, n$.

We say G is λ*-furcating* for an integer $\lambda \geqq 2$ if every $x \in S$ leads to at least λ distinct elements of S and G is *strictly* λ-furcating if every x leads to exactly λ elements. Suppose G is λ-furcating. A λ*-colouring* $\mathcal{G}$ for G is a set $\mathcal{G} = \{r_1, r_2, \cdots, r_\lambda\}$ of functions from S into S such that $x \to r_i(x)$ in G, $i = 1, 2, \cdots, \lambda$, and $r_1(x), r_2(x), \cdots, r_\lambda(x)$ are distinct for all $x \in S$. For a given λ-colouring $\mathcal{G}$, let $\mathcal{F} = \mathcal{F}(\mathcal{G}) = \mathcal{F}(r_1, r_2, \cdots, r_\lambda)$ denote the semi-group of functions $h: S \to S$ which can be written as a composition $h = h_1 h_2 \cdots h_k$ where each $h_i \in \mathcal{G}$. In the particular case $\lambda = 2$, we write *bifurcating* for 2-furcating and *colouring* for 2-colouring, and we call the functions of a colouring r and b (red and blue.).

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The *road-colouring conjecture* of Adler, Goodwyn and Weiss [1] is that for every strictly λ-furcating, irreducible, aperiodic graph G, there is a λ-colouring $\mathcal{G}$ for which $\mathcal{F}(\mathcal{G})$ contains a constant function. This conjecture arose in the course of their work in ergodic theory. The problem is also interesting from a purely graph-theoretic point of view. The principal result of this paper is that the road-colouring conjecture is true under certain circumstances which are specified in Theorem 1.

The name "road-colouring" was coined by Adler, Goodwyn and Weiss because of the following interpretation. Think of the vertices of G as cities and the edges as one-way roads between cities. Is it possible to choose two roads leading from each city and paint one red and the other blue, in such a way that a finite list of instructions of the form "follow the red road" or "follow the blue road" will lead an individual to a particular city x (after he has followed all the instructions in order), no matter what his starting point?

A *cycle* (of *length* n) in G is an ordered set $(a_1, a_2, \cdots, a_n)$ of distinct elements of S such that

$$a_n \to a_{n-1} \to \cdots \to a_1 \to a_n \quad \text{in } G. \tag{1}$$

THEOREM 1. *Let G be a finite directed graph which is irreducible, aperiodic and λ-furcating, where $\lambda \geqq 2$. Suppose G has a cycle C whose length is a prime number less than $|S|$ the cardinality of S. Then G has a λ-colouring $\mathcal{G}$ such that $\mathcal{F}(\mathcal{G})$ contains a constant function.*

Section 2 contains some preliminaries needed for the proof of Theorem 1 and Section 3 contains the proof itself. Section 4 discusses the difficulties involved in extending Theorem 1 to the case when n is not prime. The rest of the present section contains some initial observations about Theorem 1.

First, the road-colouring conjecture is easily seen to be true if G has a loop, that is, a vertex a_1 such that $a_1 \to a_1$ in G. If this is the case, let $R_1 = \{a_1\}$ and, for $i = 2, 3, \cdots$, let

$$R_i = \{y \in S - \{a_1\} \text{: the shortest path from } y \text{ to } a_1 \text{ has } i-1 \text{ edges}\}.$$

(A *path* P in G is an ordered subset $(s_0, s_1, \cdots, s_m)$ of S such that $s_{i-1} \to s_i$ in G for $i = 1, 2, \cdots, m$.) Since G is irreducible, $S = \bigcup R_i$. Now define $r(a_1) = a_1$ and for $i > 1$ and $z \in R_i$, let $r(z) = y$ where y is some element of R_{i-1} such that $z \to y$ in G. It is clear that such a y exists. Then the composition $h = rr \cdots r$ maps S into $\{a_1\}$ provided enough r's are included. Assume henceforth that G has no loops.

The problem in the general case is that if we attempt to lead occupants of all vertices to some fixed vertex by some succession of r's and b's, the first ones to arrive must depart before the later ones reach the vertex. Nevertheless, the above procedure is a key part of the proof of Theorem 1. We will use it to map S into C and subsequently to manufacture a map from C to a single point.

The assumption that G is irreducible is not essential. It is sufficient that G have a subgraph satisfying the hypotheses of Theorem 1 and that there exist a path from any vertex of G to some vertex of the subgraph, for then we may first map S to the vertex set of the subgraph, as in the case where G has a loop. If it is required that every constant function be in $\mathcal{F}(\mathcal{G})$, the irreducibility is necessary. It was shown by Adler et al. [1] that the assumption that G is aperiodic is necessary.

The conjecture as stated makes no sense without the λ-furcation assumption. A natural variation is to drop this assumption but then to broaden the concept of λ-colouring by permitting $r_i(x) = r_j(x)$ for $i \neq j$. The theorem remains true with these modifications, even without the assumption that the length of C is less than $|S|$. The proof of this is similar to the proof of Theorem 1.

The assumption that C has prime length is essential for our proof. Specifically, the conclusions of Lemmata 1 and 5 may be false without this assumption. The assumption that the length of C is less than $|S|$ is used in the proof of Lemma 3. It is only needed if G is strictly bifurcating and if each vertex is lead to by exactly two other vertices in G. If the road-colouring conjecture is true, these assumptions about C are of course unnecessary for the conclusion of Theorem 1 to hold.

The road-colouring conjecture would be false if it were modified to allow the colouring to be specified in advance. As an example, let $S = \{x, y, z\}$ and let G be the graph with edges $x \to y$, $y \to z$, $z \to x$, $x \to z$, $z \to y$ and $y \to x$. If the first three edges are coloured red and the last three blue, then r and b are both one-to-one, so that every $h \in \mathcal{F}(r, b)$ is a permutation of S. On the other hand, if the edges $x \to y$, $y \to x$ and $z \to y$ are coloured red and the others blue, then $rb(x) = r(b(x)) = r(z) = y$ and similarly $rb(y) = rb(z) = y$, so that the road-colouring conjecture is true for this graph.

2. Trees

We will use the notion of tree in the proof of Theorem 1. The reader is referred to Moon [2] or to most graph theory texts for background material. By a directed rooted tree T, we mean a tree in which the edges are all directed toward the root. We write $x \to y$ in T if y is one step below x (i.e., closer to the root) in T. The graph G and the cycle $C = (a_1, a_2, \cdots, a_n)$ will be taken as fixed throughout.

DEFINITION. A C-tree T is a directed rooted tree which satisfies the following conditions:

(i) the set of vertices of T is S,

(ii) the root of T is a_1,

(iii) if $x \to y$ in T then $x \to y$ in G, and

(iv) $a_n \to a_{n-1} \to \cdots \to a_1$ in T.

We use the following notation in connection with a C-tree T. First, let $T \cup C$ denote the directed graph whose edges are those of T together with $a_1 \to a_n$. For each $x \in S$, let $R_T(x)$ be the number of edges in T from x down to a_1, plus one. $R_T(x)$ is called the *height* of x. Let $S_T(x)$ be the number in $\{1, 2, \cdots, n\}$ such that $S_T(x) \equiv R_T(x) \pmod{n}$. Note that $S_T(a_i) = R_T(a_i) = i$ for $i = 1, 2, \cdots, n$. For $x \in S$, let $B_T(x) = \{y \in S : y \neq x$ and the path in T from y down to a_1 goes through $x\}$ and let $\bar{B}_T(x) = B_T(x) \cup \{x\}$.

If $x \in S$ and $x \to y$ in T, then $R_T(y) = R_T(x) - 1$ and $S_T(y) \equiv S_T(x) - 1 \pmod{n}$. We say $x \in S$ is *periodic* in T if, for all $y \in S$ such that $x \to y$ in G, $S_T(y) \equiv S_T(x) - 1 \pmod{n}$. If x is not periodic, we say x is *aperiodic*. For $x \neq a_1$, we say x is *R_T-constant* if, for all $y \in S$ such that $x \to y$ in G, $R_T(y) = R_T(x) - 1$; otherwise we say x is *R_T-variable.*

To clarify the above concepts, it is helpful to plot some examples, such as the following. Let G be the graph with vertex set $S = \{x, y, z, a_1, a_2, a_3\}$ and edges $a_1 \to a_3$, $a_3 \to a_2$, $a_2 \to a_1$, $a_1 \to y$, $a_2 \to y$, $a_3 \to z$, $x \to a_3$, $x \to y$, $y \to x$, $y \to z$, $z \to x$, and $z \to y$. Let C be the cycle (a_1, a_2, a_3). Let T_1 be the C-tree with edges $a_3 \to a_2$, $a_2 \to a_1$, $x \to a_3$, $y \to x$, $z \to x$. Let T_2 be the C-tree with edges $a_3 \to a_2$, $a_2 \to a_1$, $x \to a_3$, $y \to x$, $z \to y$. Observe that a_3 is periodic in T_1 but not in T_2. Also, $B_{T_1}(y) = \varnothing$ while $B_{T_2}(y) = \{z\}$. The proof of Theorem 1 is made more complicated by the possibility of two phenomena exhibited by the vertex x, namely $B_T(x) \neq \varnothing$ for *all* C-trees and $x \to a_3$ in *all* C-trees.

3. Proof of Theorem 1

We begin with an outline of the proof, which is presented in the form of several lemmata. Lemma 1 shows that it is enough to prove the theorem for the case when G is strictly bifurcating. This fact, which is then assumed for the rest of the section, greatly simplifies some of the subsequent steps. Now let T be any C-tree. Let $r(x) = y$ where $x \to y$ in $T \cup C$ and let $b(x) = z$ where $x \to z$ in G but not in $T \cup C$. The composition of a sufficiently large number of r's, say r^k, maps S onto C. Note that any two vertices x and y such that $S_T(x) = S_T(y)$ are mapped to the same element of C by r^k. Suppose there exists an $h \in \mathscr{F}(r, b)$

such that $S_T(h(a_i)) = S_T(h(a_j))$ for two distinct $a_i, a_j \in C$. Then $r^k h(a_i) = r^k h(a_j)$. It follows that $r^k h r^k$ maps S into a proper subset of C. It is shown in Lemma 5 that this is sufficient to prove the theorem. The role of Lemmata 2 and 3 is to construct a C-tree T for which a suitable h exists. The construction is broken into cases, depending on whether there exists any aperiodic elements outside C and on whether there exists an aperiodic element x for which $B_T(x) = \varnothing$. The function h itself is constructed in Lemma 4.

LEMMA 1. *It is sufficient to prove the theorem under the assumption that G is strictly bifurcating.*

PROOF. Let T be a C-tree. Since G is aperiodic, there must exist an aperiodic $x \in S$. Let G' be a strictly bifurcating graph with vertex set S and with edges chosen according to the following requirements. First suppose $x \not\in C$. Let G' contain all edges in $T \cup C$, all edges on a path leading via a non-repeating set of vertices from a_1 to x and an edge $x \to y$ where $S_T(y) \not\equiv S_T(x) - 1 \pmod{n}$. Next suppose $x \in C$. At least one element of C and all periodic elements of C lead in G to elements of $S - C$. Thus we may let G' contain all edges of $T \cup C$, and edge $a_i \to y$ where $a_i \in C$ and $S_T(y) \not\equiv i - 1$, and an edge $a_j \to z$ where $a_j \in C$, $a_j \neq a_i$ and $z \not\in C$. In either case, such a G' exists since at most two ($\leqq \lambda$) edges leading from any $u \in S$ are included in the requirements.

Now let $S'' = \{u \in S$: there is a path in G' from a_1 to $u\}$ and let G'' be the graph with vertex set S'' and with all edges of G' which lead from (and hence to) vertices in S''. G'' is evidently strictly bifurcating, irreducible and aperiodic. Its vertices include $a_1, a_2, \cdots, a_n, x, y$ and, if appropriate, z. Thus G'' has a cycle of prime length $n < |S''|$.

The proof of Lemma 1 is now completed by observing that any colouring which provides a solution to the road-colouring conjecture for G'' can be extended to a λ-colouring which provides a solution for G. We assume henceforth that G is strictly bifurcating.

The proof of Theorem 1 depends on whether a C-tree T has an aperiodic vertex outside C or the only aperiodic vertices are in C. The next result shows that all C-trees are of the same type (for a given G and C).

LEMMA 2. *Suppose a C-tree T_1 has the property that every vertex in $S - C$ is periodic. Then every C-tree T has that property, $S_T(x)$ is independent of T for all x, and the set $\{a_i \in C$: a_i is aperiodic$\}$ is independent of T.*

PROOF. The lemma is proved by induction on the number of vertices $x \in S$ such that $x \to y$ in T and $x \to z$ in T_1 where $y \neq z$. It holds when the number is 0.

Let J be the set of C-trees for which the number is at most $k-1$ and assume the statement of the lemma holds within the class J. Let T be a C-tree for which the indicated number is k. Let x be one of the indicated vertices of S such tht $R_T(x)$ is maximal, and assume $x \to y$ in T and $x \to z$ in T_1 where $y \neq z$. Let T_2 denote the graph obtained from T by deleting the edge $x \to y$ from T and adding the edge $x \to z$. Since $z \notin B_{T_1}(x)$, it follows from the maximality of $R_T(x)$ that $z \notin B_T(x)$. Therefore T_2 is a C-tree in J. By the inductive hypothesis, every vertex in $S-C$ is periodic in T_2, $S_{T_2}(s)=S_{T_1}(s)$ for all $s \in S$ and $\{a_i \in C\colon a_i$ is aperiodic in $T_2\}=\{a_i \in C\colon a_i$ is aperiodic in $T_1\}$. Since x is periodic in T_2, $S_{T_2}(y)=S_{T_2}(z)$. It follows that $S_T(s)=S_{T_2}(s)=S_{T_1}(s)$ for all $s \in S$, that all $s \in S-C$ are periodic in T and that $\{a_i \in C\colon a_i$ is aperiodic in $T\}=\{a_i \in C\colon a_i$ is aperiodic in $T_1\}$. This completes the proof of Lemma 2.

The purpose of the next lemma is to show that there exists a C-tree which has certain properties. Lemma 4 will then provide the required function h as discussed at the beginning of this section. Let T be a C-tree and let $x \in S-C$. We will be concerned with the following properties of T and x:

(P_1) x is aperiodic in T;

(P_2) every $y \in S-C$ is periodic in T and exactly one $a_i \in C$ is aperiodic in T;

(P_3) every $y \in S-C$ is periodic in T and there exist $a_i, a_j \in C$ such that

(i) $x \to a_j$ in T,

(ii) $a_i \to d$ in G where $S_T(d) \not\equiv i-1 \pmod n$, and

(iii) if $x \to v$ in G where $v \neq a_j$, then every path in $T \cup C$ from v to a_i goes through a_j;

(P_4) $B_T(x)=\varnothing$; and

(P_5) every $y \in B_T(x)$ is periodic in T and there exist $w, u, z \in \bar{B}_T(x)$ such that

(i) $B_T(w)=\varnothing$, and

(ii) $w \to z$ in T, $w \to u$ in G and $R_T(z)<R_T(u)$.

Lemma 3. *There exist a C-tree T and an $x \in S-C$ such that one of* (P_1), (P_2) *and* (P_3) *holds and one of* (P_4) *and* (P_5) *holds.*

Proof. First suppose there exists a C-tree T_1 and an aperiodic element $x \in S-C$. Choose T_1 and x from among all such C-trees and elements in such a way that $|B_{T_1}(x)|$ is as small as possible. Then (P_1) holds and it will be shown below that if (P_4) does not also hold then a modification of T_1 must also satisfy (P_5). Now suppose no such T_1 and x exist. Then some $a_i \in C$ is aperiodic. If only one element of C is aperiodic for a C-tree T then (P_2) holds and, since $n<|S|$, there exists an $x \in S-C$ such that $B_T(x)=\varnothing$ so that (P_4) also holds in this case. Using Lemma 2, the only remaining case is that every element of $S-C$ is periodic while two distinct elements a_α and a_β are aperiodic *for every C-tree*. Let $a_k \in C$ be such that $a_k \to y$ in G for some $y \in S - C$. Such an a_k exists by the irreducibility of G and the fact that $n < |S|$. Let $a_i = a_\alpha$ unless $a_\alpha = a_k$ and let $a_i = a_\beta$ otherwise. Choose $a_j \in C$ to be the *first* vertex among $a_i, a_{i+1}, \cdots, a_n, a_1, \cdots, a_{i-1}$ for which there exists $x \in S - C$ with $x \to a_j$ in G. Then choose a C-tree T_1 and $x \in S - C$ such that $x \to a_j$ in T_1 and $|B_{T_1}(x)|$ is minimized. It is easily seen that (P_3) holds in this case.

The proof of Lemma 3 is complete if (P_2) holds and, in the other two cases, it is complete if $B_{T_1}(x) = \varnothing$. Thus we need only prove the lemma under the assumption that $|B_{T_1}(x)| > 0$. If (P_1) holds for T_1, then y is periodic for all $y \in B_{T_1}(x)$ by the minimality condition on that set (otherwise, replace x by y). The same fact holds by assumption if (P_3) holds for T_1. Let $y \in B_{T_1}(x)$ and let $y \to s$ in T_1 and $y \to t$ in G where $t \neq s$. Suppose $t \notin \bar{B}_{T_1}(x)$. Then the graph T_2 obtained by deleting the edge $y \to s$ from T_1 and adding the edge $y \to t$ is a C-tree. (If t were in $B_{T_1}(y)$, this would not be the case.) By the procedure used in choosing a_j, (P_3(iii)) remains valid for T_2 in the case when (P_3) holds. Since y is periodic in T_1, it is clear that $S_{T_1}(t) = S_{T_2}(t)$ for all $t \in S$. Consequently (P_1) or (P_3) remains valid for T_2, which contradicts the minimality of $|B_{T_1}(x)|$. We conclude that if $y \in B_{T_1}(x)$ and $y \to t$ in G, then $t \in \bar{B}_{T_1}(x)$.

Now suppose more specifically that $y \to x$ in T_1 and $y \to t$ in G where $t \neq x$. By the previous argument, $t \in B_{T_1}(x)$, which implies $R_{T_1}(t) > R_{T_1}(x)$ so that y is R_{T_1}-variable. It follows that there exists $w \in B_{T_1}(x)$ such that w is R_{T_1}-variable and $R_{T_1}(w)$ is maximal. Suppose $s_1, s_2, \cdots, s_l \to w$ in T_1. Then each $s_i \to t_i$ in G for some $t_i \neq w$. Since each s_i must be R_{T_1}-constant,

$$R_{T_1}(t_i) = R_{T_1}(w), \qquad i = 1, 2, \cdots, l. \tag{2}$$

Therefore $t_i \notin B_{T_1}(s_j)$ for all i and j. The graph T_3 obtained by deleting from T_1 the edges $s_i \to w$ and adding the edges $s_i \to t_i$ for $i = 1, 2, \cdots, l$ is clearly a C-tree. By (2), $R_{T_3}(s) = R_{T_1}(s)$ for all $s \in S$ so that (P_1) and (P_3) are unaffected by the change.

It is clear that T_3 satisfies (P_5(i)). Suppose $w \to u_1$ in T_3 and $w \to u_2$ in G where $u_1 \neq u_2$ and $R_{T_3}(u_1) \neq R_{T_3}(u_2)$. If $R_{T_3}(u_2) > R_{T_3}(u_1)$, take $T = T_3$, $z = u_1$ and $u = u_2$. Otherwise, let T be the C-tree obtained by replacing the edge $w \to u_1$ by $w \to u_2$ in T_3 and let $z = u_2$ and $u = u_1$. Then (P_5) holds. This completes the proof of Lemma 3.

LEMMA 4. *There is a colouring $\{r, b\}$ and a function $H \in \mathcal{F}(r, b)$ such that the range of H is a proper subset of C.*

PROOF. The proof is split into five cases. We will give the details in one of these and sketch the proof in the other four.

Assume there is a C-tree T with vertices x, w, z and u in $S - C$ and a_i and $a_j \in C$ such that (P_3) and (P_5) both hold. Assume $i \leqq j$. Some minor variations are required if $i > j$ but, in any case, a "rotation" of C can be made to ensure that $i \leqq j$. For $s \in S$, let $r(s) = t$ where $s \to t$ in $T \cup C$ and let $b(s) = t$ where $s \to t$ in G but not in $T \cup C$. (Recall that G is assumed to be strictly bifurcating.) In particular, note that $r(x) = a_j$, $b(a_i) = d$, $r(w) = z$ and $b(w) = u$. Let H_1 be the composition of enough r's that $H_1(S) = C$. If $S_T(s) = S_T(t)$, it follows that $H_1(s) = H_1(t)$. Suppose there exists an $h \in \mathscr{F}(r, b)$ such that the set $\{S_T(h(s)): s \in C\}$ does not contain n distinct values. It follows that $|H_1 h(C)| = |H_1 h H_1(S)| < n$ which proves the lemma. We therefore assume that no such h exists. Let $P = (a_l = s_0, s_1, \cdots, s_k = w)$ be a path of minimal length in G from some $a_l \in C$ to w. Let $H_2 \in \mathscr{F}(r, b)$ be given by $H_2 = f_k f_{k-1} \cdots f_1$ where $f_\alpha = r$ if $s_{\alpha-1} \to s_\alpha \in T \cup C$ and $f_\alpha = b$ otherwise. Then $H_2(a_l) = w$. Let H_3 be the composition of $R_T(w) - i$ r's so that $H_3(w) = a_i$. Let $H = H_1 b H_3 H_2 H_1$. Since the range of H_1 is C, it is clear by our assumptions that H_2, H_3 and b can only have the effect of permuting the S_T-values when applied to C, $H_2(C)$ and $H_3H_2(C)$ respectively so that $H(C) = C$.

Now consider the modified colouring obtained by taking $r(s)$ and $b(s)$ unchanged for $s \neq w$ and taking $r(w) = u$ and $b(w) = z$. Define H as before but now assume H_1 is the composition of enough r's so that $H_1(S) = C$ for both colourings. Since w only appears as the final vertex of P and since P has minimal length, it is clear that $H_2(C)$ is the same for both colourings. Recall that $S_T(u) = S_T(z)$ but $R_T(u) > R_T(z)$. It follows that $R_T(u) \geqq R_T(z) + n$. Let $v \in S$ be $H_3(w)$ under the second colouring. Then $R_T(v) \geqq i + n > n$ so that $v \in S - C$. Therefore v is periodic and $S_T(b(v)) \equiv S_T(v) - 1$. But $S_T(b(a_i)) \not\equiv S_T(a_i) - 1$. Since $w \notin r(S)$, the vertices of $H_3H_2(C)$ are the same for both colourings except that a_i is replaced by v. It follows that $H_1 b H_3 H_2 H_1(S)$ is a proper subset of C for one of the two colourings.

We now summarize the proof in various other cases. Suppose (P_1) and (P_5) hold. Let H_3 be the composition of $R_T(w) - R_T(x)$ r's. Then the same H as defined above satisfies the required conditions. Next, suppose (P_1) and (P_4) hold. Let P be a path of minimal length from some a_l to x and let the second colouring be chosen so that $r(x)$ and $b(x)$ are reversed. Then $H_2(a_l) = x$ and $H(S) \equiv H_1 H_2 H_1(S)$ is a proper subset of C for one of the two colourings. Third, suppose (P_3) and (P_4) hold. Let H_3 be the composition of $R_T(x) - i$ r's. The definition of a_j ensures that $H_3(x) \notin C$ for the second colouring, under which $r(x) \neq a_j$. Then

$H(S) = H_1bH_3H_2H_1(S)$ is a proper subset of C for one of the two colourings. Finally, the result is obvious if (P_2) holds. By Lemma 3, all possibilities have been included, so the proof of Lemma 4 is complete.

Let $\{r, b\}$ be the colouring constructed in Lemma 4 and let $H \in \mathcal{F}(r, b)$ be such that $H(S)$ is a proper subset of C. The final step in the proof of Theorem 1 is to show that some $K \in \mathcal{F}(r, b)$ is a constant function. The next lemma shows such a K exists with K more particularly in $\mathcal{F}(r, H)$. Since $r(C) \subseteq C$ and $H(C) \subset C$, it suffices to consider the restrictions of r and of H to C.

LEMMA 5. *Let $H: C \to C$ be a function which is not one-to-one. Let $r(a_i) = a_{i-1}$, $i = 1, 2, \cdots, n$ (where $a_0 = a_n$). Then some $K \in \mathcal{F}(r, H)$ is a constant function.*

PROOF. Choose $K \in \mathcal{F}(r, H)$ such that $|K(C)|$ is minimized. Then $1 \leqq |K(C)| < n$. Let $a_\alpha, a_\beta, a_\gamma \in K(C)$ and let $a_i, a_j \in K^{-1}(a_\alpha)$. Suppose $i + \beta = j + \gamma = k$, say (calculated modulo n). Then

$$Kr^{2n+k-i-j}(a_\beta) = K(a_{\beta-2n-k+i+j}) = K(a_j) = a_\alpha$$

and, similarly,

$$Kr^{2n+k-i-j}(a_\gamma) = a_\alpha.$$

This contradicts the minimality of $|K(C)|$. Thus all elements $i + \beta$ where $a_i \in K^{-1}(a_\alpha)$ and $a_\beta \in K(C)$ are distinct, so that $|K^{-1}(a_\alpha)| \leqq n/|K(C)|$. Summing over all α such that $a_\alpha \in K(C)$, we obtain

$$n = \sum_\alpha |K^{-1}(a_\alpha)| < n,$$

unless $|K^{-1}(a_\alpha)| = n/|K(C)|$ for all $a_\alpha \in |K(C)|$. Thus $|K(C)|$ divides n. Since n is prime, $|K(C)| = 1$, i.e., K is a constant function. The proof of Lemma 5 is complete.

4. Generalizations

It is not the case that every graph G of the type specified at the beginning of this paper has a cycle of prime length, so Theorem 1 does not completely resolve the road-colouring problem.

We do not know if the road-colouring conjecture is always true, but we can make some observations about the extension of our particluar construction to more general graphs. Our colouring has the property that a cycle C is chosen in advance and all edges of C are coloured red. Such colourings do not lead to a positive answer to the road-colouring conjecture if any of the assumptions of Theorem 1 is dropped.

First, suppose the length of C is $|S|$. Consider the simple example of a graph with 3 vertices, which is given in Section 2, and let $C = (x, y, z)$. Then the unique colouring which sets $r(z) = y$, $r(y) = x$ and $r(x) = z$ cannot be used to solve the road-colouring problem. Of course, in this example, the problem can be solved if we instead take $C = (x, y)$.

If the requirement that the length of C be prime is dropped from Theorem 1, then there are cases for which no colouring $\{r, b\}$ which sets $r(a_i) = a_{i-1}$ for all $a_i \in C$ has the property that $\mathcal{F}(r, b)$ contains a constant function. For example, let $S = \{a_1, a_2, a_3, a_4, x, y\}$ and let the edges of G be $a_i \to a_{i-1}$ for all i, $x \to a_2$, $x \to a_4$, $y \to a_1$, $y \to a_3$, $a_1 \to x$, $a_3 \to x$, $a_2 \to y$, and $a_4 \to y$. It is easily seen that G satisfies the requirements of Theorem 1 with the cycle $C = \{a_1, a_2, a_3, a_4\}$ except that $n = 4$. For any C-tree T (there are only 4 possibilities), the colouring $\{r, b\}$ obtained by setting $r(s) = t$ where $s \to t$ in $T \cup C$ has the property that for any $H \in \mathcal{F}(r, b)$, $H(x) \neq H(y)$.

It may still be the case that, for all G, Theorem 1 goes through for some cycle. In the above example, $C' = (a_1, a_2, x)$ is a cycle of length 3 so Theorem 1 does solve the road-colouring problem for G.

The main problem is that Lemma 5 is only true for cycles of prime length. In general, the best one can say is that there is a non-empty class $\mathcal{F}_1$ of functions from C to C such that some composition of the function and r's gives a constant function. If n is prime, $\mathcal{F}_1$ contains all functions which are not one-to-one. We have obtained some information about the class $\mathcal{F}_1$ in [3], but have not characterized it in any useful way. An attempt to generalize Theorem 1 might involve a characterization of $\mathcal{F}_1$ for composite n, followed by a strengthening of Lemma 4 to produce a function H whose restriction to C is in $\mathcal{F}_1$. The above example shows that it would be necessary to choose C appropriately as well. Thus, generalization is likely to be quite difficult.

References

1. R. L. Adler, L. W. Goodwyn and B. Weiss, *Equivalence of topological Markov shifts*, Israel J. Math. **27** (1977), 49–63.

2. J. W. Moon, *Counting Labelled Trees*, Canadian Mathematical Congress Monograph No. 1, 1970.

3. G. L. O'Brien, *Zero-inducing functions on finite abelian groups*, to appear (1981).

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